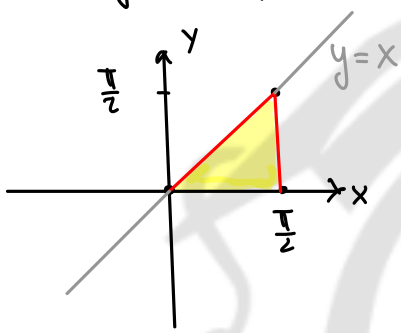
VEČKRATNI INTEGRALI

DVOJNI INTEGRAL

$$\boxed{I = \int_D f(x,y)\,dx\,dy}$$

Integral je enak volumnu telesa, ki ga omejuje območje D, graf funkcije f in vse navpičnice v robnih točkah območja D.

PRIMER: Izračunaj integral funkcije $f(x,y) = \cos x \cdot \cos y$ po trikotniku z oglišči $(0,0)$, $(\frac{\pi}{2},0)$, $(\frac{\pi}{2},\frac{\pi}{2})$

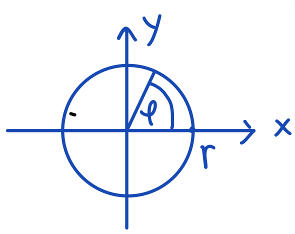


$$I = \int_0^{\pi/2}\left(\int_0^x \cos x\cdot\cos y\, dy\right)dx =$$
$$= \int_0^{\pi/2} \cos x\cdot \sin y \Big|_0^x dx =$$
$$= \int_0^{\pi/2} \cos x\cdot \sin x\, dx =$$
$$= \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

nova spremenljivka
$t = \sin x$
$dt = \cos x\,dx$

Polarne koordinate

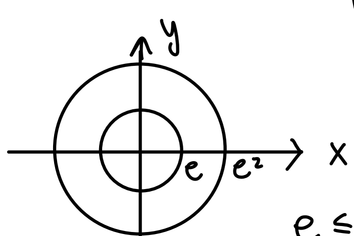
– uporabimo jih kadar integriramo krog ali del kroga.



$$\boxed{\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array} \quad J = r}$$

$$\boxed{\int_D f(x,y)\,dx\,dy = \int_K f(x\cos\varphi, y\sin\varphi)\, r\,dr\,d\varphi}$$

PRIMER: Izračunaj $\iint_D \ln(x^2+y^2)\,dx\,dy$, kjer je D kolobar med krožnicama $x^2+y^2=e^2$ in $x^2+y^2=e^4$



$e \le r \le e^2$
$0 \le \varphi \le 2\pi$

$x = r\cos\varphi$, $y = r\sin\varphi$

$$x^2+y^2 = r^2\cos^2\varphi + y^2\sin^2\varphi =$$
$$= r^2(\cos^2\varphi+\sin^2\varphi) =$$
$$= r^2$$

$$\boxed{\cos^2\varphi + \sin^2\varphi = 1}$$

$$I = \int_0^{2\pi}\int_e^{e^2} \ln(r^2)\cdot r\,dr\,d\varphi = 2\pi\cdot\left(\frac{r^2}{2}\ln(r^2)\Big|_e^{e^2} - \int_e^{e^2} r\,dr\right) = 2\pi\left(\frac{e^4}{2}\cdot 4 - \frac{e^2}{2}\cdot 2 - \frac{r^2}{2}\Big|_e^{e^2}\right) =$$

PER PARTES $u = \ln(r^2) \longrightarrow du = \frac{1}{r^2}\cdot 2r\,dr = \frac{2}{r}dr$

$dv = r\,dr \longrightarrow v = \frac{r^2}{2}$

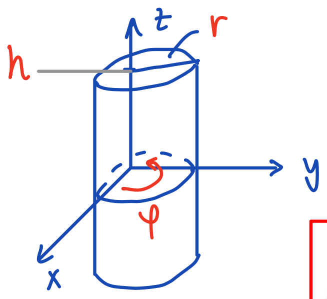
$$= 2\pi\left(2e^4 - e^2 - \frac{e^4}{2} + \frac{e^2}{2}\right) =$$
$$= 2\pi\left(\frac{3e^4}{2} - \frac{e^2}{2}\right) = \underline{\underline{\pi(3e^4 - e^2)}}$$

TROJNI INTEGRAL

$$\boxed{I = \int_D f(x,y,z)\,dxdydz}$$

Cilindrične koordinate

– uporabimo jih kadar integriramo po valju - cilindru ali podobnemu telesu



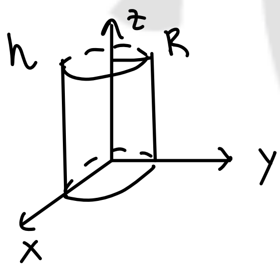
$$\boxed{\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned} \qquad J = r}$$

$$\boxed{\int_{D_1} f(x,y,z)\,dxdydz = \int_{D_2} f(r\cos\varphi, r\sin\varphi, z)\cdot r\,d\varphi\,dr\,dz}$$

PRIMER: G je valj s polmerom R, osnovno ploskvijo na xy ravnini, ki ima za os kar os z, višina pa je h.

$$G = \{(x,y,z);\ x^2+y^2 \le R^2, 0 \le z \le h\}$$

Izračunaj: $I = \int_G (x^2+y^2+z^2)\,dxdydz$



$$0 \le r \le R$$
$$0 \le \varphi \le 2\pi$$
$$0 \le z \le h$$

$$I = \int_0^{2\pi}\int_0^{h}\int_0^{R} (r^2+z^2)\cdot r\,dr\,d\varphi\,dz =$$

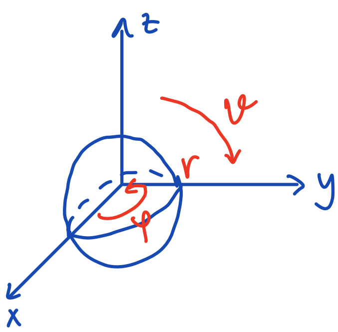
$$= 2\pi \int_0^h \int_0^R (r^3 + rz^2)\,dr\,dz = 2\pi \int_0^h \left(\frac{r^4}{4} + \frac{r^2z^2}{2}\right)\Big|_0^R dz =$$

$$= 2\pi \int_0^h \left(\frac{R^4}{4} + \frac{R^2z^2}{2}\right) dz = 2\pi\left(\frac{R^4}{4}z + \frac{R^2}{2}\cdot\frac{z^3}{3}\right)\Big|_0^h =$$

$$= 2\pi\left(\frac{R^4h}{4} + \frac{R^2h^3}{6}\right) = \underline{\underline{\frac{R^4h\pi}{2} + \frac{R^2h^3\pi}{3}}}$$

Sferične / krogelne koordinate

– uporabimo jih kadar integriramo po krogli, delu krogle ali podobnem telesu

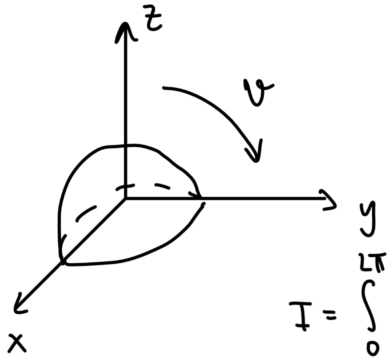


$$\boxed{\begin{aligned} x &= r\cos\varphi\sin\vartheta \\ y &= r\sin\varphi\sin\vartheta \\ z &= r\cos\vartheta \end{aligned} \qquad J = r^2\sin\vartheta}$$

$$\boxed{\int_{D_1} f(x,y,z)\,dxdydz = \int_{D_2} f(r\cos\varphi\sin\vartheta, r\sin\varphi\sin\vartheta, r\cos\vartheta)\,r^2\sin\vartheta\,dr\,d\varphi\,d\vartheta}$$

PRIMER: Izračunajte $\int_D z\,dx\,dy\,dz$, kjer je D zgornja polovica krogle s središčem v izhodišču in polmerom R.

$$D = \{(x,y,z)\,;\, x^2+y^2+z^2 \le R^2 \,;\, z \geq 0\}$$



$$0 \le r \le R$$
$$0 \le \varphi \le 2\pi$$
$$0 \le \vartheta \le \frac{\pi}{2}$$

$$I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R r\cos\vartheta \cdot r^2 \sin\vartheta \; d\varphi\, d\vartheta\, dr =$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos\vartheta \sin\vartheta \, d\vartheta \int_0^R r^3\, dr =$$

$t = \sin\vartheta$

$dt = \cos\vartheta\, d\vartheta$

$$= 2\pi \int_0^1 t\, dt \cdot \frac{r^4}{4}\Big|_0^R = 2\pi \frac{t^2}{2}\Big|_0^1 \frac{R^4}{4} = \cancel{2}\pi \cdot \frac{1}{\cancel{2}} \cdot \frac{R^4}{4} = \underline{\underline{\frac{\pi R^4}{4}}}$$